Bending of a Square Plate With Two Adjacent Edges Free and the Others Clamped or Simply Supported

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(1) Introduction

THE LITERATURE abounds with applications of classical thin elastic plate theory to rectangular plates having a multitude of combinations of boundary conditions and transverse loading. By far the greatest number of these papers deal with plates having only simply-supported and/or clamped boundary conditions, which requires restrictions upon the deflection and its first or second derivatives along the edges. Considerably fewer solutions have been obtained when a free edge is involved because of the relatively difficult combination of third derivatives of deflection which is encountered in the boundary conditions. When two adjacent free edges are involved, so that a free corner is created, the problem becomes exceedingly difficult. No closed form of solution is known to exist for any of this latter class of problems.

Some solutions have been obtained for cantilevered rectangular plates—i.e., three edges free and the other clamped. Finite difference solutions for various aspect ratios and loadings were presented by Holl,[1] Barton,[2] MacNeal,[3] Nash,[4] and Livesly and Birchall.[5] Nash[4] also solved the problem of the uniformly loaded cantilevered plate having a span-to-chord ratio of 1/2 by using a form of collocation known as point matching. This method depends upon choosing deflection functions which satisfy the partial differential equation of the continuum exactly, while matching the boundary conditions at only a finite number of discrete points. Nash[4] solved the problem twice, once using an algebraic polynomial and, again, using a hyperbolic-trigonometric series. A recent paper by Leissa and Niedenfuhr[6] presents the solution for the uniformly loaded cantilevered square plate using two other approaches: point matching, using an algebraic-trigonometric polynomial, and a Rayleigh-Ritz minimal-energy formulation.

Doubtlessly, much of the foregoing work with cantilevered plates has been the result of the need for accurate information for the design of aircraft wings. However, approximate deflections and stresses may be obtained for these problems by using elementary beam theory. In marked contrast, the rectangular plate having two adjacent edges free and the others clamped or simply supported is in no way representable by beam theory. These problems are no more difficult than the cantilevered plate, but very little has been done with them up to the present. The problem of the uniformly loaded square plate with two adjacent edges free and the others clamped was solved by Huang and Conway.[7] This involved a skillful superposition of five problems and the partial solution of an infinite set of simultaneous equations. Yeh[8] generalized this same problem by including the reaction due to an elastic foundation. The Rayleigh-Ritz method was employed.

In the present work the writers exhibit solutions for the four problems of a square plate with two adjacent edges free and the others clamped or simply supported subjected to either a uniform transverse loading or a concentrated force at the free corner. The method of point matching is used to solve the problems and the results are compared with other known solutions for two of the cases.

(2) Analysis

Consider the constant-thickness plate of arbitrary shape shown in Fig. 1. It will be assumed that segments of the boundary may be either simply-supported, clamped, or free. The polar coordinates r and θ are used to locate points on the boundary and within the region of the plate in terms of an arbitrarily placed origin. The angle ϕ is used to define direction at any point on the plate. In the case of a point on the boundary, ϕ measures the angle by which the outer normal to the boundary leads the radial line drawn from the coordinate origin.

The differential equation to be solved is

$$\nabla^4 w = q/D \quad (1)$$

where ∇^4 is the customary biharmonic operator, w the deflection, q the transverse loading, and D the flexural rigidity of the plate. One solution to this equation is

$$w = \sum_{m=0}^{\infty} (A_m r^m \cos m\theta + B_m r^m \sin m\theta + C_m r^{m+2} \cos m\theta + D_m r^{m+2} \sin m\theta) + w_p \quad (2)$$

where the infinite series is the complementary solution and w_p is the particular solution of Eq. (1). This form

Received by IAS April 12, 1962. Revised and received October 30, 1962.

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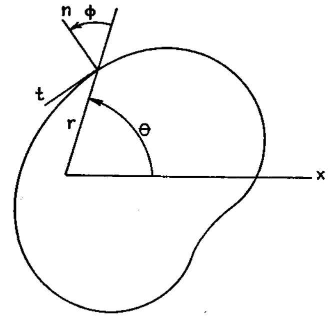

FIG. 1. Plate coordinates.

of solution was first used with the point matching method by Barta[9] on the bending of a square plate, and by Conway[10] who discussed bending of square and triangular plates clamped along all boundaries. To determine slopes, moments, and transverse shears within the plate and on the boundary, combinations of first, second, and third derivatives of Eq. (2) must be used. The derivation of these quantities is presented in detail in Ref. 6.

(3) Solutions to Problems

The method was applied to the solution of the following four problems pertaining to the bending of a square plate:

(1) Uniformly loaded, two adjacent edges clamped and the others free.

(2) Uniformly loaded, two adjacent edges simply supported and the others free.

(3) Concentrated load at the free corner, two adjacent edges simply supported and the others free.

(4) Concentrated load at the free corner, two adjacent edges clamped and the others free.

Fig. 2 shows the orientation of the coordinate system

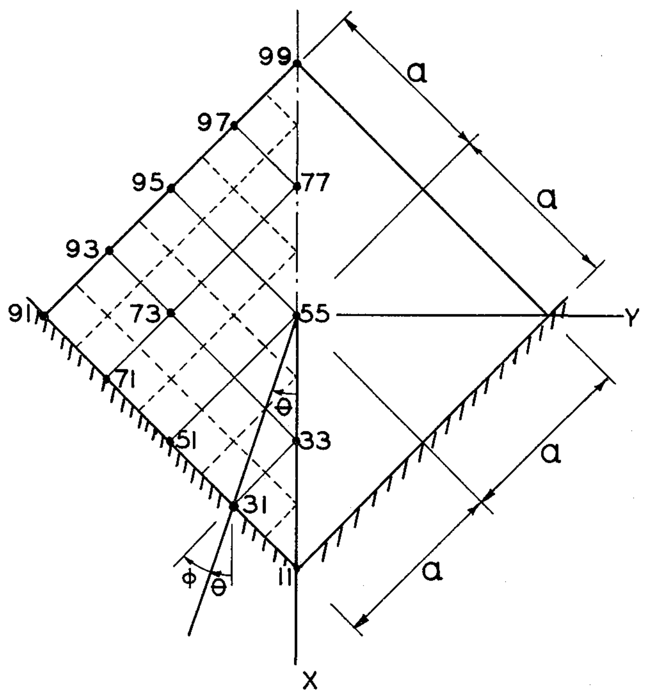

FIG. 2. Location and identification of points.

relative to the plate. A square grid of width 1/8 the plate dimensions is superimposed to aid in describing locations of points on the boundary and in the interior of the plate. A two digit numbering system is used, the digits increasing along the sides of the plate as indicated. In this manner, points 11, 21, 31, 41, 51, 61, 71, 81, 91 occur along a clamped or simply supported edge and points 91, 92, 93, 94, 95, 96, 97, 98, 99 are found along a free edge, with point 99 being the free corner. Because of the symmetry of the four problems to be solved, only the even deflection functions in-

TABLE 1. Solutions for Coefficients

	Uniformly Loaded		Concentrated Force in Free Corner	
coefficient	clamped - free	S. S. - free	S. S. - free	clamped - free
A_o	1.3399×10^{-1}	9.7371×10^{-1}	7.4999×10^{-1}	1.2238×10^{-1}
C_oa^2	-4.5680×10^{-3}	-1.1198×10^{-1}	-2.8273×10^{-7}	7.2938×10^{-2}
A_1a	-2.5839×10^{-1}	-1.2297	-1.0606	-3.1559×10^{-1}
C_1a^3	3.1353×10^{-2}	2.3043×10^{-2}	1.6253×10^{-7}	-1.4066×10^{-3}
A_2a^2	1.2434×10^{-1}	3.8607×10^{-1}	3.7500×10^{-1}	1.8354×10^{-1}
C_2a^4	-2.1561×10^{-2}	-9.7688×10^{-5}	1.0482×10^{-7}	-1.9854×10^{-2}
A_3a^3	-5.2045×10^{-3}	4.8051×10^{-2}	8.3989×10^{-7}	-4.2003×10^{-2}
C_3a^5	-9.9020×10^{-3}	-4.2118×10^{-3}	-7.2345×10^{-8}	-3.7157×10^{-3}
A_4a^4	-6.6046×10^{-3}	-8.2693×10^{-3}	-6.8806×10^{-8}	4.8472×10^{-4}
C_4a^6	-5.3044×10^{-5}	-9.3760×10^{-4}	-2.4117×10^{-10}	2.4278×10^{-3}
A_5a^5	2.1738×10^{-3}	3.6111×10^{-3}	-1.7303×10^{-7}	-7.7723×10^{-4}
C_5a^7	7.6903×10^{-4}	-1.7895×10^{-4}	6.0495×10^{-8}	1.5037×10^{-3}
A_6a^6	2.1049×10^{-4}	-2.3579×10^{-4}	-3.5438×10^{-8}	5.8327×10^{-4}
C_6a^8	-1.1943×10^{-4}	7.7567×10^{-5}	2.8488×10^{-8}	-1.4391×10^{-4}
A_7a^7	5.4009×10^{-4}	1.6578×10^{-4}	8.3453×10^{-8}	3.1955×10^{-4}
C_7a^9	-1.8085×10^{-4}	-5.2315×10^{-5}	-4.7002×10^{-8}	-2.0805×10^{-4}
A_8a^8	-6.1699×10^{-5}	1.3675×10^{-5}	1.2639×10^{-7}	-1.4048×10^{-4}
C_8a^{10}	1.0532×10^{-4}	-1.7234×10^{-5}	-8.5978×10^{-8}	8.0198×10^{-5}
A_9a^9	-1.4940×10^{-4}	2.7024×10^{-5}	1.0085×10^{-7}	-2.1522×10^{-4}
C_9a^{11}	5.7834×10^{-5}	-1.1186×10^{-5}	-6.3472×10^{-8}	8.7622×10^{-5}
$A_{10}a^{10}$	1.0851×10^{-4}	2.0168×10^{-5}	5.8010×10^{-9}	3.4795×10^{-5}
$C_{10}a^{12}$	-9.9397×10^{-5}	-2.2857×10^{-5}	7.6368×10^{-10}	-3.3317×10^{-5}
$A_{11}a^{11}$	1.3806×10^{-4}	4.7220×10^{-5}	-9.5290×10^{-8}	7.8278×10^{-5}
$C_{11}a^{13}$	-7.2070×10^{-5}	-1.9734×10^{-5}	5.7490×10^{-8}	-4.2600×10^{-5}
$A_{12}a^{12}$	-2.5276×10^{-5}	9.6139×10^{-6}	-1.0386×10^{-7}	-8.3840×10^{-6}
$C_{12}a^{14}$	2.9982×10^{-5}	-1.6065×10^{-6}	5.7362×10^{-8}	8.8882×10^{-6}
$A_{13}a^{13}$	-1.1988×10^{-4}	-3.5948×10^{-5}	-4.3639×10^{-8}	-5.0841×10^{-5}
$C_{13}a^{15}$	7.7404×10^{-5}	1.8614×10^{-5}	2.2769×10^{-8}	3.1132×10^{-5}
$A_{14}a^{14}$	-9.5420×10^{-5}	-5.3422×10^{-5}	6.0619×10^{-9}	-2.4529×10^{-5}
$C_{14}a^{16}$	5.8933×10^{-5}	2.9628×10^{-5}	-3.1305×10^{-9}	1.5295×10^{-5}
$A_{15}a^{15}$	3.1438×10^{-5}	3.5253×10^{-6}	2.9110×10^{-8}	1.5192×10^{-5}
$C_{15}a^{17}$	-1.8574×10^{-5}	-5.8879×10^{-6}	-1.4894×10^{-8}	-5.8774×10^{-6}
$A_{16}a^{16}$	3.3106×10^{-5}	1.6912×10^{-5}	1.3228×10^{-8}	8.7768×10^{-6}
$C_{16}a^{18}$	-2.2283×10^{-5}	-9.3924×10^{-6}	-7.2479×10^{-9}	-5.8785×10^{-6}
$A_{17}a^{17}$	-2.7647×10^{-6}	4.2953×10^{-6}	-8.4735×10^{-10}	-3.8993×10^{-6}

volving cos $n\theta$ are needed from the complimentary solution of Eq. (2). Accordingly, boundary conditions are only required at points along two sides of the plate.

For each of the four problems a total of 35 conditions were applied at discrete points around the boundary. When dealing with a clamped edge the constraints of zero deflection and normal slope were applied at each of the 9 points 11 to 91. No constraints were placed

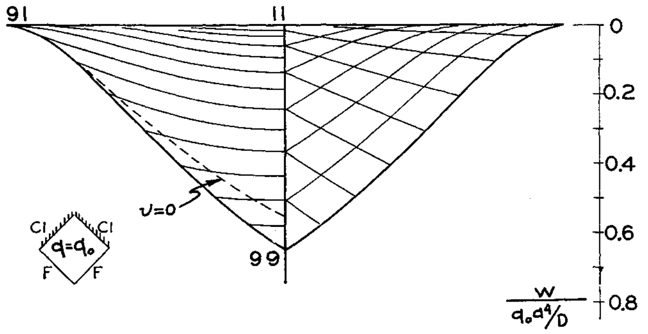

FIG. 3. Deflected shape. Two adjacent edges clamped, others free. Uniformly loaded.

upon the tangential slopes. For a simply supported edge the conditions of zero deflection and normal moment were used at these points. In the case of the free edge no normal moment or edge reaction was allowed to occur at points 92 to 99. Because of the singularity of the moments and shears in the corners, the free edge conditions were not enforced at point 91. Indeed, it was found that very unrealistic solutions were obtained when these latter conditions were added.

According to the accepted Kelvin-Kirchoff theory it was necessary to add the extra condition of zero twisting moment at the free corner if no concentrated reaction was desired at this point, as in the case of the first two problems. Conversely, in accordance with the theory, a concentrated force P was applied at the free corner in the last two problems, by requiring the twisting moment to be $1/2P$ there.

Thus, for each of the problems a set of 35 simultaneous linear algebraic equations was formulated in the 35 undetermined coefficients A_0, C_0, . . . , A_{17} by substituting the values of r, θ, and ϕ for each point appropriately into the equations expressing the boundary conditions at the point. Poisson's ratio of 1/3 was used throughout. In the case of the uniform loading the particular solutions became the numerical right-hand sides of the simultaneous equations. When using the concentrated free corner force only, the algebraic equations are all homogeneous, with the exception of the one boundary-condition equation which specifies the corner force. The solutions of these sets of equations for the 35 coefficients are presented in Table 1.

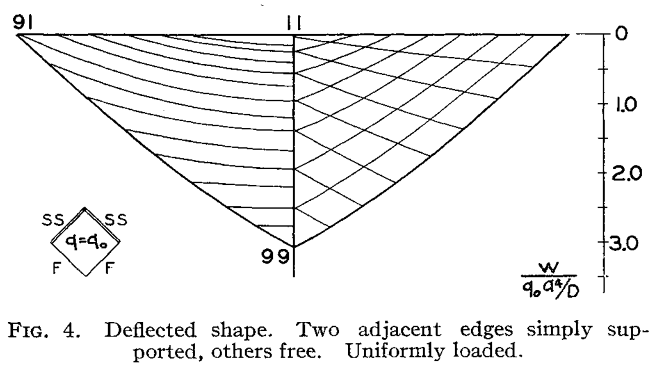

FIG. 4. Deflected shape. Two adjacent edges simply supported, others free. Uniformly loaded.

Knowing the coefficients, the problems are solved, and useful information, particularly the deflections and moments, may be found at desired points within the regions. In this way, deflections and moments were evaluated at each of the 45 points shown in Fig. 2.

In Figs. 3 and 4 the deflections $w/(q_0a^4/D)$ are plotted for the uniformly loaded cases. Fig. 5 shows the deflections $w/(Pa^2/D)$ calculated for the case having a concentrated force in the free corner and two edges clamped. The deflections are viewed as one would see them looking in the plane of the plate along the axis of symmetry. On the left side of the symmetry axes curves are drawn which show the deformations of lines normal to the symmetry axis—e.g., the line joining points 92, 83, 74, and 65. The deflected pattern of the orthogonal grid lines is shown to the right of the symmetry axis. One additional curve is shown dotted on Fig. 3. This is the deflection of the free edge for $\nu = 0$. No figure is drawn for the case of the point load on the free corner and simply supported for this solution is well known and discussed in Section 4 of the paper.

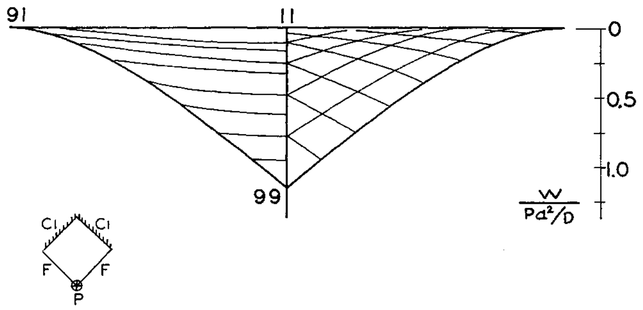

FIG. 5. Deflected shape. Two adjacent edges clamped, others free. Concentrated force in free corner.

In this last case the free edges remain straight lines when deflected.

For purposes of having accurate reference values for possible future comparisons, calculated deflections are presented in Table 2 to five-place accuracy for several significant points. Because of accumulated roundoff error, particularly when solving the 35 simultaneous equations, this is the limit of dependable accuracy obtainable from the electronic digital computer when single precision arithmetic (8 significant figures) is used to solve problems of this size.

Bending moments were calculated at all 45 points in the directions parallel to the diagonals of the plate—i.e., Mx and My—in terms of the coordinates shown in Fig. 2. These moments are tabulated in Tables 3 and 4. Because of space considerations, values are given only at alternate points. Moments were also computed at the 9 points along the clamped edges in the direction normal to the edge. Their values are presented in Table 5.

TABLE 2. Deflections at Significant Points

	Uniformly Loaded		Concentrated Force in Free Corner	
Point	clamped - free	S. S. - free	S. S. - free	clamped - free
33	1.6029 x 10^{-2} $q_o a^4/D$	2.6794 x 10^{-1} $q_o a^4/D$	1.8750 x 10^{-1} Pa^2/D	6.9915 x 10^{-3} Pa^2/D
55	1.3399 x 10^{-1}	9.7371 x 10^{-1}	7.5000 x 10^{-1}	1.2238 x 10^{-1}
73	7.7158 x 10^{-2}	7.2646 x 10^{-1}	1.8750 x 10^{-1}	6.9924 x 10^{-3}
77	3.6548 x 10^{-1}	1.9569	1.6875	4.8529 x 10^{-1}
95	3.0351 x 10^{-1}	1.7731	1.5000	3.8471 x 10^{-1}
99	6.4873 x 10^{-1}	3.0841	3.0000	1.1591

TABLE 3. Bending Moment, M_x

	Uniformly Loaded		Concentrated Force in Free Corner	
Point	clamped - free	S. S. - free	S. S. - free	clamped - free
11	+1.7989 x 10^{-2} $q_o a^2$	-7.7935 x 10^{-1} $q_o a^2$	-5.0000 x 10^{-1} P	-5.1533 x 10^{-3} P
31	-1.1555 x 10^{-1}	-7.04641 x 10^{-1}	"	-7.2806 x 10^{-3}
33	-1.8651 x 10^{-1}	-5.1294 x 10^{-1}	"	-1.9597 x 10^{-1}
51	-3.4090 x 10^{-1}	-6.2198 x 10^{-1}	"	-1.6744 x 10^{-1}
53	-2.3893 x 10^{-1}	-3.7783 x 10^{-1}	"	-3.5435 x 10^{-1}
71	-5.5709 x 10^{-1}	-5.7628 x 10^{-1}	"	-4.4548 x 10^{-1}
55	-1.5360 x 10^{-1}	-2.1615 x 10^{-1}	"	-4.3922 x 10^{-1}
73	-2.8683 x 10^{-1}	-3.1073 x 10^{-1}	"	-1.9600 x 10^{-1}
91	-6.8117 x 10^{-1}	-5.1841 x 10^{-1}	"	-6.3826 x 10^{-1}
75	-1.0820 x 10^{-1}	-1.3971 x 10^{-1}	"	-4.7903 x 10^{-1}
93	-3.1619 x 10^{-1}	-3.5599 x 10^{-1}	"	-4.8584 x 10^{-1}
77	-1.7841 x 10^{-2}	-4.6349 x 10^{-2}	"	-4.8403 x 10^{-1}
95	-1.3287 x 10^{-1}	-1.7510 x 10^{-1}	"	-4.7581 x 10^{-1}
97	-1.9509 x 10^{-2}	-5.5417 x 10^{-2}	"	-4.7876 x 10^{-1}
99	0	0	"	-5.0000 x 10^{-1}

(4) Comparison of the Results with Known Solutions

Although the effort required to set up and solve large sets of simultaneous equations becomes too great for hand calculation methods on problems of the type considered here, very satisfactory results can be obtained by using an electronic digital computer. As mentioned above each of the 4 problems were treated by solving 35 simultaneous equations using 8 significant figures and single precision arithmetic. The choice of 35 equations was one of convenience and was not dictated by machine limitations.

Of the four problems discussed here, one has an "exact" solution in closed form—viz., the plate simply supported on two edges and carrying a point load at the free corner. This problem meets the same boundary conditions as the square plate with a uniform twisting moment induced by four concentrated corner loads. The solution of this problem is well known, and in terms of the coordinate system of Fig. 2 may be written

$$w = \frac{Pa}{D}\left[-\frac{3}{4} + \frac{3\sqrt{2}}{4}\left(\frac{r}{a}\right)\cos\theta - \frac{3}{8}\left(\frac{r}{a}\right)^2\cos 2\theta\right] \quad (3)$$

Making use of 35 coefficients we find by the point matching method:

$$w = \frac{Pa}{D}\left[-0.74999297 + 1.060497\left(\frac{r}{a}\right)\cos\theta - 0.37499535\left(\frac{r}{a}\right)^2\cos 2\theta\right] \quad (4)$$

while the other 32 coefficients were found to be smaller than 10^{-6} as is indicated in Table 1. The differences between this and the exact solution in the sixth significant figure are due to machine round-off error. This adequately represents to all intents and purposes the exact solution of this problem, and as such lends validity to the other three solutions obtained in this paper. It should be emphasized that no special precautions were taken with this test problem as far as its set up and introduction into the machine are concerned.

Caution should be used when attempting to apply this exact solution to a real problem, however. The solution actually pertains to a plate under the influence of twisting moments distributed along the edges and not really to a plate with concentrated corner loads. (The stresses σ_{xy} and deflection w predicted by plate theory are the same as predicted for this problem by St. Venant's torsion theory for thin bars.) The deflection function of Eq. (3) may be observed to be a harmonic function, so that the transverse shears are identically zero. In the case of the real plate simply supported on

TABLE 4. Bending Moment, M_y

	Uniformly Loaded		Concentrated Force in Free Corner	
Point	clamped - free	S. S. - free	S. S. - free	clamped - free
11	-6.7216 x 10^{-2} $q_o a^2$	7.7935 x 10^{-1} $q_o a^2$	+5.0000 x 10^{-1} P	-5.1463 x 10^{-2} P
31	-1.1446 x 10^{-1}	7.0323 x 10^{-1}	"	-1.0725 x 10^{-2}
33	5.1863 x 10^{-2}	8.0945 x 10^{-1}	"	-1.9170 x 10^{-2}
51	3.3366 x 10^{-1}	6.2352 x 10^{-1}	"	-1.6790 x 10^{-1}
53	4.3525 x 10^{-2}	7.9518 x 10^{-1}	"	-3.1219 x 10^{-2}
71	-5.2525 x 10^{-1}	5.9209 x 10^{-1}	"	-4.4317 x 10^{-1}
55	1.7821 x 10^{-1}	8.1336 x 10^{-1}	"	5.0215 x 10^{-2}
73	-2.8658 x 10^{-2}	7.4713 x 10^{-1}	"	-1.9192 x 10^{-2}
91	3.9021 x 10^{-2}	1.0794	"	-5.5853 x 10^{-1}
75	1.8546 x 10^{-1}	7.4697 x 10^{-1}	"	9.4652 x 10^{-2}
93	4.4336 x 10^{-4}	7.1998 x 10^{-1}	"	-5.6568 x 10^{-2}
77	2.1429 x 10^{-1}	6.2441 x 10^{-1}	"	2.0741 x 10^{-1}
95	1.8476 x 10^{-1}	4.7942 x 10^{-1}	"	1.5722 x 10^{-1}
97	1.6668 x 10^{-1}	3.8844 x 10^{-1}	"	3.0264 x 10^{-1}
99	0	0	"	5.0000 x 10^{-1}

TABLE 5. Bending Moments Along Clamped Edges, M_n

Point	Uniformly Loaded	Concentrated Force in Free Corner
11	-2.4614 x 10^{-2} $q_o a^2$	-2.8308 x 10^{-2} P
21	-3.0878 x 10^{-2}	2.1297 x 10^{-2}
31	-1.7327 x 10^{-1}	-1.4145 x 10^{-2}
41	-3.3951 x 10^{-1}	-1.0737 x 10^{-1}
51	-5.0799 x 10^{-1}	-2.5198 x 10^{-1}
61	-6.8487 x 10^{-1}	-4.4381 x 10^{-1}
71	-8.3013 x 10^{-1}	-6.7032 x 10^{-1}
81	-1.1563	-9.9882 x 10^{-1}
91	-1.1836	-1.0897

two adjacent edges and loaded at its free corner by a concentrated load there are, of course, nonzero shears. In spite of this obvious deficiency, the exact solution is known to correspond very closely to experimental results as far as deflections and twisting moments are concerned.

The uniformly loaded square plate with adjacent edges clamped can be compared numerically with the solution by Huang and Conway.[7] These authors superimpose five classical plate bending problems to achieve their solution, and while they do not give extensive numerical results, we may at least compare deflections at the free corner. For Poisson's ratio of zero, Huang and Conway obtain

$$w_{\max} = 0.57904\, q_0a^4/D \tag{5}$$

while the point matching procedure gives (for $\nu = 0$) a value of

$$w_{\max} = 0.55915\, q_0a^4/D \tag{6}$$

a difference of less than 3.5 percent. When Poisson's ratio is 1/3 we obtain

$$w_{max} = 0.64873\, q_0a^4/D \tag{7}$$

which is about a 15 percent increase over the previous value. Evidently the effects of changing Poisson's ratio are far from negligible in this problem.

When studying the four solutions presented here one cannot help but be struck by the large influence of clamping moments on the deflection. Comparing maximim deflection for the uniformly loaded cases, we see from Table 2 that $w_{max} = 3.0841q_0a^4/D$ in the simply supported case, which is 4.77 times as large as the clamped edge case carrying the same load. Similarly the plate with simply supported edges carrying a concentrated load at the tip yields $w_{max} = 3Pa^2/D$ compared to $w_{max} = 1.1591Pa^2/D$ when two edges are clamped. The comparison at point 33 is even more pronounced.

The computing time by IBM 704 required to set up and solve the 35 equations, then evaluate and tabulate deflection, 2 slopes, 2 bending moments, twisting moment, and shear at each grid point was approximately the same for each of the 4 problems treated here, being on the order of 2 min and 20 sec. No specific convergence studies were run on these problems since that question seems to have had sufficient experimental study in Reference.[6]

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